

THE VOPĚNKA PRINCIPLE IS INEQUIVALENT TO BUT CONSERVATIVE OVER THE VOPĚNKA SCHEME

JOEL DAVID HAMKINS

ABSTRACT. The Vopěnka principle, which asserts that every proper class of first-order structures in a common language admits an elementary embedding between two of its members, is not equivalent over GBC to the first-order Vopěnka scheme, which makes the Vopěnka assertion only for the first-order definable classes of structures. Nevertheless, the two Vopěnka axioms are equiconsistent and they have exactly the same first-order consequences in the language of set theory. Specifically, GBC plus the Vopěnka principle is conservative over ZFC plus the Vopěnka scheme for first-order assertions in the language of set theory.

The *Vopěnka principle* is the assertion that for every proper class \mathcal{M} of first-order \mathcal{L} -structures, for a set-sized language \mathcal{L} , there are distinct members of the class $M, N \in \mathcal{M}$ with an elementary embedding $j : M \rightarrow N$ between them. In quantifying over classes, this principle is a single assertion in the language of second-order set theory, and it makes sense to consider the Vopěnka principle in the context of a second-order set theory, such as Gödel-Bernays set theory GBC, whose language allows one to quantify over classes. In this article, GBC includes the global axiom of choice.

In contrast, the first-order *Vopěnka scheme* makes the Vopěnka assertion only for the first-order definable classes \mathcal{M} (allowing set parameters, and henceforth in this article, by the term “definable class,” I shall intend that parameters are allowed). This theory can be expressed as a scheme of first-order statements, one for each possible definition of a class, and it makes sense to consider the Vopěnka scheme in Zermelo-Frankel ZFC set theory with the axiom of choice.

Because the Vopěnka principle is a second-order assertion, it does not make sense to refer to it in the context of ZFC set theory, whose first-order language does not allow quantification over classes; one typically retreats to the Vopěnka scheme in that context. The theme of this article is to investigate the precise meta-mathematical interactions between these two treatments of Vopěnka’s idea.

Main Theorems.

- (1) *If ZFC and the Vopěnka scheme holds, then there is a class forcing extension, adding classes but no sets, in which GBC and the Vopěnka scheme holds, but the Vopěnka principle fails.*
- (2) *If ZFC and the Vopěnka scheme holds, then there is a class forcing extension, adding classes but no sets, in which GBC and the Vopěnka principle holds.*

I would like to thank Victoria Gitman for helpful conversations about the Vopěnka principle. My research has been supported by grant #69573-00 47 from the CUNY Research Foundation. Commentary can be made on the author’s blog at <http://jdh.hamkins.org/vopenka-principle-vopenka-scheme>.

These results appear as theorems 11 and 12, respectively. It follows that the Vopěnka principle VP and the Vopěnka scheme VS are not equivalent, but they are equiconsistent and indeed, they have the same first-order consequences.

Corollaries.

- (1) *Over GBC, the Vopěnka principle and the Vopěnka scheme, if consistent, are not equivalent.*
- (2) *Nevertheless, the two Vopěnka axioms are equiconsistent over GBC.*
- (3) *Indeed, the two Vopěnka axioms have exactly the same first-order consequences in the language of set theory. Specifically, GBC plus the Vopěnka principle is conservative over ZFC plus the Vopěnka scheme for assertions in the first-order language of set theory.*

$$\text{GBC} + \text{VP} \vdash \phi \quad \text{if and only if} \quad \text{ZFC} + \text{VS} \vdash \phi$$

These consequences are explained in corollaries 13 and 14.

1. A -EXTENDIBLE CARDINALS

It turns out that the Vopěnka principle and the Vopěnka scheme admit a convenient large-cardinal characterization in terms of the class-relativized extendible cardinals, and so let us develop a little of that large-cardinal theory here, before giving the characterization in section 2.

Namely, we define that a cardinal κ is *extendible*, if for every ordinal $\lambda > \kappa$ there is an ordinal θ and an elementary embedding $j : V_\lambda \rightarrow V_\theta$ with critical point κ and $\lambda < j(\kappa)$. Every such cardinal is, of course, a measurable cardinal, and indeed, a supercompact cardinal, since we may define the induced normal fine measures by $X \in \mu \leftrightarrow j \restriction \beta \in j(X)$ for $X \subseteq P_\kappa \beta$, provided $\beta + 1 < \lambda$.

More generally, we say that a cardinal κ is *A -extendible* for a class A , if for every ordinal $\lambda > \kappa$ there is an ordinal θ and an elementary embedding

$$j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$$

with critical point κ and $\lambda < j(\kappa)$. Let us refer to λ as the *degree* of A -extendibility of this embedding. Let's show next that the $\lambda < j(\kappa)$ requirement is an inessential convenience.

Lemma 1. *A cardinal κ is A -extendible if and only if for every ordinal $\lambda > \kappa$ there is an ordinal θ and an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$ with critical point κ .*

Proof. The point here is that in the latter property, we have dropped the requirement that $\lambda < j(\kappa)$. For the case of extendible cardinals, where there is no predicate A , this equivalence is proved in Kanamori's book [Kan04, proposition 23.15], and one may simply carry his argument through here with the predicate A . But let me give an alternative slightly simpler argument. The forward implication is immediate. Conversely, suppose that κ is A -extendible with respect to the weaker property, and consider any ordinal $\lambda > \kappa$. What I claim is that for every ordinal α , there is an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$ with critical point κ and $j(\kappa) > \alpha$. The case $\alpha = \lambda$ then proves the lemma. I shall prove the claim by induction on α . The statement is clearly true for all ordinals α up to the next inaccessible cardinal above κ , since $j(\kappa)$ will be at least that large regardless. Suppose by induction that the statement is true for all $\beta < \alpha$, where $\kappa < \alpha$. Let $\bar{\lambda}$ be larger

than λ and $\alpha + 2$ and large enough so that the embeddings witnessing the induction assumption all exist inside $V_{\bar{\lambda}}$. By A -extendibility, there is an ordinal θ and an elementary embedding $j : \langle V_{\bar{\lambda}}, \in, A \cap V_{\bar{\lambda}} \rangle \rightarrow \langle V_{\theta}, \in, A \cap V_{\theta} \rangle$ with critical point κ . It cannot be that α is closed under j , meaning that $j \restriction \alpha \subseteq \alpha$, since if this were true then the critical sequence would be entirely below α , and the supremum of the critical sequence would be a fixed point of j below or at α , contrary to the Kunen inconsistency. So there is some ordinal β with $\beta < \alpha < j(\beta)$. By the choice of $\bar{\lambda}$, there is inside $V_{\bar{\lambda}}$ an elementary embedding $h : \langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle \rightarrow \langle V_{\eta}, \in, A \cap V_{\eta} \rangle$ with critical point κ and $h(\kappa) > \beta$. Since this embedding is an element of $V_{\bar{\lambda}}$, we have $\eta < \bar{\lambda}$, and so the composition $j \circ h$ makes sense. Since h is elementary from $\langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle$ to $\langle V_{\eta}, \in, A \cap V_{\eta} \rangle$ and $j \restriction V_{\eta}$ is elementary from $\langle V_{\eta}, \in, A \cap V_{\eta} \rangle$ to $\langle V_{j(\eta)}, \in, A \cap V_{j(\eta)} \rangle$, it follows that $j \circ h$ is an elementary embedding from $\langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle$ to $\langle V_{j(\eta)}, \in, A \cap V_{j(\eta)} \rangle$, with critical point κ . Since $\beta < h(\kappa)$, it follows that $j(\beta) < j(h(\kappa)) = (j \circ h)(\kappa)$. Since $\alpha < j(\beta)$, we have therefore verified the desired property for α . By induction, therefore, every ordinal α has this property, and so in particular, there is an elementary embedding witnessing the A -extendibility of κ for which $j(\kappa) > \lambda$, as desired. \square

Note that if $j : \langle V_{\lambda+1}, \in, A \cap V_{\lambda+1} \rangle \rightarrow \langle V_{\theta+1}, \in, A \cap V_{\theta+1} \rangle$ is an A -extendibility embedding of degree $\lambda + 1$, then by extracting the induced extender embedding and applying that embedding to all of V , we may produce an extender ultrapower embedding $j : V \rightarrow M$ which agrees with the original embedding j on V_{λ} . In particular, we'll have critical point κ , $\lambda < j(\kappa)$, $V_{j(\lambda)} \subseteq M$ and $j(A \cap V_{\lambda}) = A \cap V_{\theta}$. This provides another characterization of A -extendibility.

If κ is A -extendible, then let us define that a set $g \subseteq \kappa$ is *A -stretchable*, if for every $\lambda > \kappa$ and every $h \subseteq \lambda$ for which $h \cap \kappa = g$, there is an elementary embedding $j : \langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle \rightarrow \langle V_{\theta}, \in, A \cap V_{\theta} \rangle$ with critical point κ and $\lambda < j(\kappa)$, for which $j(g) \cap \lambda = h$. Thus, an A -stretchable set g is one that can be stretched by an A -extendibility embedding so as to agree with any desired h extending g .

Theorem 2. *If κ is A -extendible for some class A , then there is an A -stretchable set $g \subseteq \kappa$.*

Proof. We define the initial segments of g in stages. Fix a well-ordering \triangleleft of V_{κ} . If $g \cap \gamma$ is defined, then let $\lambda > \gamma$ be least such that there is some $h \subseteq \lambda$ extending $g \cap \gamma$, such that there is no elementary embedding $j : \langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle \rightarrow \langle V_{\theta}, \in, A \cap V_{\theta} \rangle$ with critical point γ and $\lambda < j(\gamma)$, for which $j(g \cap \gamma) \cap \lambda = h$. That is, $g \cap \gamma$ does not anticipate h with respect to any A -extendibility embedding of degree λ . An easy reflection argument shows that $\lambda < \kappa$. For this minimal λ , let h be the \triangleleft -least such set that is not anticipated, and define $g \restriction \lambda = h$. This procedure defines g all the way up to κ .

To see that g is A -stretchable, suppose that there is some λ and set $h \subseteq \lambda$ extending g for which there is no A -extendibility embedding $j : \langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle \rightarrow \langle V_{\theta}, \in, A \cap V_{\theta} \rangle$ with $j(g) \cap \lambda = h$. Let $\bar{\lambda} > \lambda$ be large enough so that this assertion about λ, κ, g, h is absolute to $\langle V_{\bar{\lambda}}, \in, A \cap V_{\bar{\lambda}} \rangle$. By A -extendibility, there is an ordinal $\bar{\theta}$ and an elementary embedding $j : \langle V_{\bar{\lambda}}, \in, A \cap V_{\bar{\lambda}} \rangle \rightarrow \langle V_{\bar{\theta}}, \in, A \cap V_{\bar{\theta}} \rangle$ with critical point κ and $\bar{\lambda} < j(\kappa)$. The set $j(g) \subseteq j(\kappa)$ is defined by the same procedure as g , except using $j(\triangleleft)$, which is a well-order of $V_{j(\kappa)}$. At stages below κ , of course this procedure agrees with what happened in the construction of g . At stage κ , since $\bar{\lambda} \leq \theta$, the model will agree that λ is least for which g has no A -extension

to some set $h' \subseteq \lambda$ extending g , and so it will select the $j(\triangleleft)$ -least such h' and define $j(g) \cap \lambda = h'$. So $\langle V_{\bar{\theta}}, \in, A \cap V_{\bar{\theta}} \rangle$ will believe that there is no A -extendibility embedding for κ of degree λ , for which g anticipates h' . But this contradicts the fact that $j \upharpoonright V_{\lambda}$ is precisely such an embedding, and it is an element of $V_{\bar{\theta}}$. Therefore, g must be A -stretchable, as desired. \square

If one were to make a bounded number of changes to g below κ , then it would not affect the stretchability feature, and consequently every bounded subset of κ can be extended to an A -stretchable set g . I shall make use of this observation in the proof of theorem 12.

One should look upon stretchability as a form of the Laver diamond property for A -extendibility. (See [Ham02] for diverse versions of the Laver diamond for various large cardinals.) Specifically, if κ is A -extendible, then a function $\ell : \kappa \rightarrow V_{\kappa}$ is an A -extendibility *Laver function*, if for every λ and every target a , there is an elementary embedding $j : \langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle \rightarrow \langle V_{\theta}, \in, A \cap V_{\theta} \rangle$ with critical point κ and $\lambda < j(\kappa)$, for which $j(\ell)(\kappa) = a$. Essentially the same argument as in theorem 2 establishes:

Theorem 3. *If κ is A -extendible, then there is an A -extendibility Laver function $\ell : \kappa \rightarrow V_{\kappa}$.*

Proof. This can be seen as an immediate consequence of theorem 2, simply by coding the object a as a set of ordinals, and decoding ℓ from the stretchable set g .

Alternatively, we may also undertake a direct argument. Namely, if $\ell \upharpoonright \gamma$ is defined, then let λ be least such that some object a is not anticipated by any A -extendibility embedding $j : \langle V_{\lambda}, \in, A \cap V_{\lambda} \rangle \rightarrow \langle V_{\theta}, \in, A \cap V_{\theta} \rangle$ with critical point γ . Let $\ell(\gamma)$ be the least such a with respect to a fixed well-ordering \triangleleft of V_{κ} . A reflection argument shows that $\lambda < \kappa$ and $a \in V_{\kappa}$. If the resulting $\ell : \kappa \rightarrow V_{\kappa}$ does not have the Laver function property, let λ be least such that there is some a that is not anticipated by any A -extendibility embedding of degree λ . Let $\bar{\lambda} > \lambda$ be large enough so that this property is seen in $V_{\bar{\lambda}}$, and fix an elementary embedding $j : \langle V_{\bar{\lambda}}, \in, A \cap V_{\bar{\lambda}} \rangle \rightarrow \langle V_{\bar{\theta}}, \in, A \cap V_{\bar{\theta}} \rangle$ with critical point κ and $\bar{\lambda} < j(\kappa)$. In $V_{\bar{\theta}}$, the definition of $j(\ell)(\kappa)$ will find the same λ and pick some a' not anticipated by any A -extendibility embedding of degree λ . But this is contradicted by the existence of $j \upharpoonright V_{\lambda}$ itself, which exists in $V_{\bar{\theta}}$. \square

Corollary 4. *In GBC, for any class A there is a class function $\ell : \text{Ord} \rightarrow V$, such that whenever κ is A -extendible, then $\ell \upharpoonright \kappa$ is an A -extendible Laver function for κ .*

Proof. The point is that a global well-ordering \triangleleft provides a uniform method to define the Laver function ℓ all the way up to Ord . \square

If one does not have global choice, then one still can construct at least an *ordinal-anticipating* Laver function $\ell : \text{Ord} \rightarrow \text{Ord}$, such that for any A -extendible cardinal, the function $\ell \upharpoonright \kappa$ has the Laver function property as far as anticipating ordinals is concerned. In particular, this function will have the A -extendibility *Menas* property.

Although I won't require this in the proof of the main theorems, let me anyway investigate the degree of reflectivity and correctness for A -extendible cardinals. An ordinal γ is $\Sigma_n(A)$ -correct, if $\langle V_{\gamma}, \in, A \cap V_{\gamma} \rangle \prec_{\Sigma_n} \langle V, \in, A \rangle$. A cardinal κ is $\Sigma_n(A)$ -reflecting, if it is inaccessible and $\Sigma_n(A)$ -correct.

Theorem 5. *If κ is A -extendible for a class A , then κ is $\Sigma_2(A)$ -reflecting. If κ is $A \oplus C$ -extendible, where C is the class of all $\Sigma_1(A)$ -correct ordinals, then κ is $\Sigma_3(A)$ -reflecting.*

Proof. First, let's notice that every A -extendible κ is $\Sigma_1(A)$ -reflecting. Upward absoluteness is immediate. Conversely, suppose that there is some x for which $\varphi(x, a)$, where all quantifiers are bounded (but there is a predicate for A). The witness x exists in some V_λ , and so $\langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi(x, a)$. Let $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$ witness the A -extendibility of κ , with $\lambda < j(\kappa)$. Thus, $x \in V_{j(\kappa)}$, and so $\langle V_{j(\kappa)}, \in, A \cap V_{j(\kappa)} \rangle$ satisfies $\exists x \varphi(x, a)$. By elementarity, therefore, this is also true in $\langle V_\kappa, \in, A \cap V_\kappa \rangle$, as desired.

Also every A -extendible cardinal is $\Sigma_2(A)$ -reflecting. Again, the upward absoluteness is immediate by the previous paragraph. If there is some x for which $\forall y \varphi(x, y, a)$, where φ has only bounded quantifiers (and A may appear as a predicate, then pick λ large enough so that $x \in V_\lambda$, and again consider the embedding j as above. Note again that $x \in V_\lambda \subseteq V_{j(\kappa)}$. And since $\varphi(x, y, a)$ hold for all y , we see that $\langle V_{j(\kappa)}, \in, A \cap V_{j(\kappa)} \rangle \models \exists x \forall y \varphi(x, y, a)$, and so again this pulls back to $\langle V_\kappa, \in, A \cap V_\kappa \rangle$ by elementarity, as desired.

Finally, consider $\Sigma_3(A)$, and suppose now that κ is $A \oplus C$ -extendible, where C is the class of $\Sigma_1(A)$ -correct ordinals. The upward absoluteness of $\Sigma_3(A)$ from $\langle V_\kappa, \in, A \cap V_\kappa \rangle$ follows from the $\Sigma_2(A)$ -correctness of κ established in the previous paragraph. So suppose that $\exists x \forall y \exists z \varphi(x, y, z, a)$ holds in $\langle V, \in, A \rangle$ for some $a \in V_\kappa$. Let λ be large enough so that the witness x is in V_λ . Let $j : \langle V_\lambda, \in, A \cap V_\lambda, C \cap \lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta, C \cap \theta \rangle$ witness the $A \oplus C$ -extendibility of κ . Since $\kappa \in C$, it follows that $j(\kappa) \in C$ also, and so $j(\kappa)$ is $\Sigma_1(A)$ -correct. Thus, since $x \in V_\lambda \subseteq V_{j(\kappa)}$ and $\langle V, \in, A \rangle \models \forall y \exists z \varphi(x, y, z, a)$, it follows that $\langle V_{j(\kappa)}, \in, A \cap V_{j(\kappa)} \rangle \models \forall y \exists z \varphi(x, y, z, a)$. Consequently, this model satisfies the assertion $\exists x \forall y \exists z \varphi(x, y, z, a)$, and so we may pull back the statement to $\langle V_\kappa, \in, A \cap V_\kappa \rangle$, thereby verifying this instance of the $\Sigma_3(A)$ -correctness of κ , as desired. \square

Note that without the predicate A , every \beth -fixed point is Σ_1 -correct, since this is the Lévy reflection theorem. Thus, when there is no A , we get for free that $j(\kappa)$ is Σ_1 -correct, and so every extendible cardinal is Σ_3 -reflecting. It is not clear to me at the moment in the general case whether every A -extendible cardinal is $\Sigma_3(A)$ -reflecting, since the argument appears to rely on the $\Sigma_1(A)$ -correctness of $j(\kappa)$.

2. LARGE-CARDINAL CHARACTERIZATION OF VOPĚNKA'S PRINCIPLE

I shall now give the large-cardinal characterization of the Vopěnka principle.

Theorem 6. *The following are equivalent over $\text{GB} + \text{AC}$ set theory.*

- (1) *The Vopěnka principle.*
- (2) *For every class A and all sufficiently large λ , there is an ordinal $\theta > \lambda$ and an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$.*
- (3) *For every class A , there is an A -extendible cardinal.*
- (4) *For every class A , there is a stationary proper class of A -extendible cardinals.*
- (5) *For every class A and all sufficiently large ordinals λ , there is a transitive class M and an elementary embedding $j : V \rightarrow M$ with some critical point $\kappa < \lambda$, such that $\lambda < j(\kappa)$ and $j(A \cap V_\lambda) = A \cap V_{j(\lambda)}$.*

Proof. (1 \rightarrow 2) Assume towards contradiction that the Vopěnka principle holds, but statement 2 fails for some class A . So there are unboundedly many ordinals λ for which there is no elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$ for any ordinal $\theta > \lambda$. Let \mathcal{M} be the class of all such structures $\langle V_\lambda, \in, A \cap V_\lambda \rangle$ having no such embedding. By the Vopěnka principle, there is an elementary embedding between two of these structures $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$, with $\lambda < \theta$, contrary to the inclusion of the former structure in \mathcal{M} .

(2 \rightarrow 3) Suppose that A is a class and for some ordinal λ_0 and all $\lambda \geq \lambda_0$, there is an ordinal $\theta > \lambda$ and an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$. For singular λ , we may assume without loss that j has a critical point below λ , by considering $j \upharpoonright V_\lambda$ for an embedding j on $V_{\lambda+1}$, which must move λ , but cannot have λ as its critical point. So we have a critical point $\kappa < \lambda$, although different λ could have different such critical points. Nevertheless, the map $\lambda \mapsto \kappa$, choosing the smallest such κ , is a definable pressing-down function, and so by the class version of Fodor's lemma, there is a stationary class of λ for which the embeddings all have the same critical point κ . Thus, this constant value κ is the critical point of elementary embeddings $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$ for unboundedly many ordinals λ . By restricting these embeddings, it follows that κ is the critical point of such embedding for every $\lambda > \kappa$, and so κ is A -extendible by lemma 1.

(3 \rightarrow 4) Suppose that for every class A , there is an A -extendible cardinal. Fix any class A and and class club C . It suffices to find an A -extendible cardinal κ in C . Let $A \oplus C$ be a class coding both A and C , and let κ be $A \oplus C$ -extendible. Let $\lambda > \kappa$ be at least as large as the next element of C above κ . Since κ is $A \oplus C$ -extendible, there is an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda, C \cap \lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta, C \cap \theta \rangle$, with critical point κ and $j(\kappa) > \lambda$. Since $C \cap j(\kappa)$ has elements above κ , it follows that C is not bounded below κ , and so $\kappa \in C$, as desired.

(4 \rightarrow 1) Assume statement 4, and suppose that \mathcal{M} is a proper class of structures in a first-order language \mathcal{L} . Let κ be an \mathcal{M} -extendible cardinal above the size of the language \mathcal{L} . Let λ be any ordinal above κ for which there is an element $M \in \mathcal{M}$ of rank λ . Since κ is \mathcal{M} -extendible, there is an ordinal θ and an elementary embedding $j : \langle V_{\lambda+1}, \in, \mathcal{M} \cap V_{\lambda+1} \rangle \rightarrow \langle V_{\theta+1}, \in, \mathcal{M} \cap V_{\theta+1} \rangle$ with critical point κ and $\lambda < j(\kappa)$. Note that $j(\lambda) = \theta$ and so $\lambda < \theta$. Since $j(M) \in \mathcal{M}$ is a structure of rank θ in \mathcal{M} , it is therefore not identical with M . Since the language is fixed pointwise by j , it follows that $j \upharpoonright M : M \rightarrow j(M)$ is an elementary embedding between distinct elements of \mathcal{M} , thus verifying this instance of the Vopěnka principle.

(3 \leftrightarrow 5) The forward implication is immediate by the remarks after the proof of lemma 1. For the converse, fix any class A , and apply statement 5 to the class $A \oplus V = (\{0\} \times A) \cup (\{1\} \times V)$. Thus, for all sufficiently large ordinals λ there is a transitive class M and nontrivial elementary embedding $j : V \rightarrow M$ with critical point $\kappa < \lambda$ and $\lambda < j(\kappa)$, and the final condition of statement 5 for the class $A \oplus V$ amounts to $j(A \cap V_\lambda) = A \cap V_{j(\lambda)}$ and $j(V_\lambda) = V_{j(\lambda)}$. This latter condition implies $V_{j(\lambda)} \subseteq M$ and so κ is A -extendible simply by restricting the embedding to V_λ . \square

Essentially identical arguments work in ZFC with the first-order Vopěnka scheme, by considering only definable classes:

Theorem 7. *The following are equivalent over ZFC set theory.*

- (1) *The Vopěnka scheme.*

- (2) For every definable class A and all sufficiently large λ , there is an ordinal $\theta > \lambda$ and an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$.
- (3) For every definable class A , there is an A -extendible cardinal.
- (4) For every definable class A , there is a definably stationary proper class of A -extendible cardinals.
- (5) For every definable class A and all sufficiently large ordinals λ , there is a definable transitive class M and a definable nontrivial elementary embedding $j : V \rightarrow M$ with critical point below λ , such that $j(A \cap V_\lambda) = A \cap V_{j(\lambda)}$.

One may also extract a version of the theorem for Vopěnka cardinals. Namely, a cardinal δ is a *Vopenka cardinal*, if δ is inaccessible and for every set $\mathcal{M} \subseteq V_\delta$ of δ many first-order \mathcal{L} -structures, with \mathcal{L} of size less than δ , there are structures $M \neq N$ in \mathcal{M} with an elementary embedding $j : M \rightarrow N$ between them. This is equivalent to saying that the structure $\langle V_\delta, \in, V_{\delta+1} \rangle$, a model of Kelley-Morse set theory whose sets are the elements of V_δ and whose classes include all subsets of V_δ , is a model of the Vopěnka principle. For example:

Theorem 8. *The following are equivalent, for any inaccessible cardinal δ .*

- (1) δ is a Vopěnka cardinal.
- (2) For every $A \subseteq V_\delta$ and all sufficiently large $\lambda < \delta$, there is an ordinal θ with $\lambda < \theta < \delta$ and an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$.
- (3) For every $A \subseteq V_\delta$, there is a $(<\delta, A)$ -extendible cardinal. That is, there is $\kappa < \delta$ such that for every λ with $\kappa < \lambda < \delta$, there is an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$, with critical point κ and $\lambda < j(\kappa)$.
- (4) For every $A \subseteq V_\delta$, there is a stationary set of such $(<\delta, A)$ -extendible cardinals below δ .

The proof simply follows the same argument as in theorem 6.

As an analogue of the Vopěnka scheme at this level, let us define that δ is a *Vopěnka-scheme cardinal*, if V_δ satisfies ZFC plus the Vopěnka scheme. The difference is whether the classes are definable or not and also whether inaccessibility is required. We similarly get a list of equivalents in analogy with theorem 7.

Theorem 9. *The following are equivalent for any cardinal δ .*

- (1) δ is a Vopěnka-scheme cardinal.
- (2) For every $A \subseteq V_\delta$ definable in $\langle V_\delta, \in \rangle$ and all sufficiently large $\lambda < \delta$, there is an ordinal $\theta < \delta$ and an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$.
- (3) For every $A \subseteq V_\delta$ definable in $\langle V_\delta, \in \rangle$, there is a $(<\delta, A)$ -extendible cardinal. That is, there is $\kappa < \delta$ such that for every λ with $\kappa < \lambda < \delta$, there is an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$, with critical point κ and $\lambda < j(\kappa)$.
- (4) For every such definable $A \subseteq V_\delta$, there is a definably stationary set of such $(<\delta, A)$ -extendible cardinals below δ .

Let us briefly call attention to a consequence of statement 4 in each of the theorems we have proved in this section. Namely, the Vopěnka principle implies that Ord is Mahlo, meaning that every class club $C \subseteq \text{Ord}$ contain a regular cardinal, since indeed, every such club contains an A -extendible cardinal, for any class A ,

and such cardinals are supercompact and more. Similarly, the Vopěnka scheme implies that Ord is definably Mahlo, the scheme asserting that every definable class club $C \subseteq \text{Ord}$ contains a regular cardinal, and indeed, it contains an A -extendible cardinal for any definable class A . The same idea shows that every Vopěnka cardinal is Mahlo, and every Vopěnka-scheme cardinal is definably Mahlo.

Corollary 10. *Every Vopěnka cardinal δ has a club set of Vopěnka scheme cardinals below δ . In particular, there is a stationary set of inaccessible Vopěnka scheme cardinals below δ , and indeed, a stationary set of $(<\delta, A)$ -extendible Vopěnka scheme cardinals below δ , for any particular $A \subseteq V_\kappa$.*

Proof. Suppose that δ is a Vopěnka cardinal. The collection of $\gamma < \delta$ for which $V_\gamma \prec V_\delta$ is club in δ . Since any instance of the Vopěnka scheme is first-order expressible, it follows that all such γ are Vopěnka scheme cardinals. And since by theorem 8 statement 4, the collection of $(<\delta, A)$ -extendible cardinals below δ is stationary, by intersecting with the club it follows that there is a stationary set of $(<\delta, A)$ -extendible Vopěnka scheme cardinals below δ . \square

In particular, the existence of a Vopěnka cardinal is strictly stronger in consistency strength over ZFC than the existence of a Vopěnka scheme cardinal, and indeed, stronger than an extendible Vopěnka scheme cardinal. Since the Vopěnka cardinals are analogous to the Vopěnka principle in the way that Vopěnka scheme cardinals are analogous to the Vopěnka scheme, this might suggest that the Vopěnka principle should be stronger in consistency strength than the Vopěnka scheme. But that conclusion would be incorrect, as the main results of this article show that GBC plus the Vopěnka principle is in fact equiconsistent with and indeed conservative over ZFC plus the Vopěnka scheme. For this reason, it is more correct to say that Vopěnka cardinals are analogous to Kelley-Morse KM set theory plus the Vopěnka principle than to GBC plus the Vopěnka principle, and the former theory is strictly stronger than the latter, for essentially similar reasons as in the proof of corollary 10.

3. SEPARATING THE PRINCIPLE FROM THE SCHEME

Let us now establish the first part of the main theorem by proving theorem 11, which shows that the two Vopěnka axioms, if consistent, are not equivalent. This result was part of my answer [Ham10] to a question posted on MathOverflow by Mike Shulman, who had inquired whether there would always be a definable counterexample to the Vopěnka principle, whenever it should happen to fail. I interpret the question as asking whether the Vopěnka scheme is necessarily equivalent to the Vopěnka principle, and the answer is negative.

Theorem 11. *If the Vopěnka scheme holds, then there is a class forcing extension $V[G]$, not adding sets, in which the Vopěnka scheme continues to hold, but the Vopěnka principle fails.*

Proof. Work in GBC set theory, and assume that the Vopěnka scheme holds. Force by initial segments to add a club $C \subseteq \text{Ord}$ avoiding the regular cardinals, and let $V[C]$ be the GBC model arising as the forcing extension. For every cardinal λ , the collection of conditions reaching above λ is $\leq \lambda$ -closed, and so this forcing adds no new sets. Thus, the sets of the forcing extension $V[C]$ are exactly the sets in V , but the classes are those definable in the structure $\langle V, \in, C \rangle$. Since the

first-order definable classes (using only set parameters) are exactly the same in $V[C]$ as in V , we have an A -extendible cardinal for any such definable class A in $V[C]$. Consequently, the first-order Vopěnka scheme continues to hold in $V[C]$. Meanwhile, since the class C destroys “Ord is Mahlo” with respect to the new non-definable classes, such as C itself, it follows by the observations at the end of the previous section that the Vopěnka principle fails in $V[C]$, as desired. \square

A similar argument applies to the definability-stratification of the Vopěnka scheme into definable levels. Specifically, by adding a class club avoiding the $C^{(n)}$ -cardinals, one can force the failure of the Σ_{n+1} -Vopěnka scheme, while preserving the Σ_n -Vopěnka scheme.

4. CONSERVATIVITY OF THE PRINCIPLE OVER THE SCHEME

In this final section, I’d like to prove the main conservativity result, namely, that the Vopěnka principle is conservative over the Vopěnka scheme for first-order statements in the language of set theory. It follows that the two Vopěnka axioms are equiconsistent over GBC.

Theorem 12. *If ZFC and the Vopěnka scheme holds, then there is a class forcing extension, adding classes but no sets, in which GBC and the Vopěnka principle holds.*

Proof. Assume that ZFC and the Vopěnka scheme holds in $\langle V, \in \rangle$. Let us force the global axiom of choice by adding a V -generic Cohen class of ordinals $G \subseteq \text{Ord}$, using the class forcing $\text{Add}(\text{Ord}, 1)$, whose conditions are elements of $2^{<\text{Ord}}$, each describing a possible initial segment of G . The class G is V -generic in the sense of meeting every definable dense subclass of this forcing. Since the forcing is κ -closed for every cardinal κ , the forcing extension $V[G]$ has the same sets as V . The classes of $V[G]$ are those definable in the structure $\langle V, \in, G \rangle$, allowing the predicate G . It is well-known that this is a model of GBC, and indeed this is how one proves that GBC is conservative over ZFC.

Let me start by proving that there is a G -extendible cardinal in $V[G]$. First, note that in V there is a stationary proper class of extendible cardinals. By density, there must be an extendible cardinal κ for which $G \cap \kappa$ is stretchable, since by theorem 2 and the remarks after it, any given condition can be extended to a stretchable set at any desired extendible cardinal above that condition. I claim that this κ is G -extendible in $V[G]$, and in fact, the condition $g = G \cap \kappa$ will force that κ is G -extendible. To see this, consider any $\lambda > \kappa$ and any stronger condition p extending g . By making λ larger, I may assume without loss that $p \subseteq \lambda$. By stretchability, there is an elementary embedding $j : V_{\lambda+1} \rightarrow V_{\theta+1}$ with critical point κ and $j(\kappa) > \lambda$ for which $j(g) \cap \lambda = p$. Let $q = j(g)$ and let $r = j(q \cap \lambda)$. Thus, $j \upharpoonright V_\lambda : \langle V_\lambda, \in, q \cap \lambda \rangle \rightarrow \langle V_\theta, \in, r \rangle$ is an elementary embedding with critical point κ and $\lambda < j(\kappa)$. But note also—and this is the key point—that since $q \cap \lambda$ extends g , we have that $j(q \cap \lambda)$ extends $j(g)$, which is q . So r extends q , and in particular, $r \cap \lambda = q \cap \lambda$. So we may write the elementary embedding as $j \upharpoonright V_\lambda : \langle V_\lambda, \in, r \cap \lambda \rangle \rightarrow \langle V_\theta, \in, r \cap \theta \rangle$. Thus, the condition r forces that κ is G -extendible at least to degree λ . But r extended p , and so we have proved that for ordinal $\lambda > \kappa$, it is dense that κ is G -extendible to degree λ . So by genericity, κ is fully G -extendible.

We may now easily augment the previous argument with any class A that is first-order definable in $\langle V, \in \rangle$, in order to find an $A \oplus G$ -extendible cardinal κ in $V[G]$. Namely, we first find an A -extendible cardinal κ for which $g = G \cap \kappa$ is A -stretchable. This exists by genericity, using theorem 2 and the fact that there are unboundedly many A -extendible cardinals. Now argue that g forces that κ is $A \oplus G$ -extendible in $V[G]$. For any condition p extending g and ordinal $\lambda > \kappa$, we may assume $p \subseteq \lambda$ and then by A -stretchability find an elementary embedding $j : \langle V_{\lambda+1}, \in, A \cap V_{\lambda+1} \rangle \rightarrow \langle V_{\theta+1}, \in, A \cap V_{\theta+1} \rangle$ with critical point κ and $\lambda < j(\kappa)$, for which $j(g) \cap \lambda = p$. Let $q = j(g)$ and $r = j(q \cap \lambda)$. So again we have $r \cap \lambda = q \cap \lambda$ and $j \upharpoonright V_\lambda : \langle V_\lambda, \in, A \cap V_\lambda, r \cap \lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta, r \cap \theta \rangle$ is an elementary embedding with critical point κ and $\lambda < j(\kappa)$. So r forces that κ is $A \oplus G$ -extendible to degree λ . So there are a dense class of such conditions and therefore by genericity, the cardinal κ is $A \oplus G$ -extendible to every degree.

The argument until now considered only classes that were definable in the ground model and G itself. So now let us finally consider the case where A is a class that is definable from G in $\langle V, \in, G \rangle$. Suppose that $A = \{a \mid V[G] \models \varphi(a, z, G)\}$ for some first-order formula φ , in which a predicate for G may appear, and parameter z . By the definability of the forcing relation, we know that the classes $\dot{A} = \{\langle a, p \rangle \mid p \Vdash \varphi(a, z, \dot{G})\}$ and $\bar{A} = \{\langle a, p \rangle \mid p \Vdash \neg \varphi(a, z, \dot{G})\}$ are definable in the ground model V . By the previous paragraph, there is a cardinal κ that is $(\dot{A} \oplus \bar{A} \oplus G)$ -extendible. Fix any λ , and let $\bar{\lambda} > \lambda$ be large enough so that for every $a \in V_\lambda$, there is a condition $p = G \cap \alpha$ with $\alpha < \bar{\lambda}$ that decides $\varphi(a, z, \dot{G})$. Let

$$j : \langle V_{\bar{\lambda}}, \in, \dot{A} \cap V_{\bar{\lambda}}, \bar{A} \cap V_{\bar{\lambda}}, G \cap \bar{\lambda} \rangle \rightarrow \langle V_{\bar{\theta}}, \in, \dot{A} \cap V_{\bar{\theta}}, \bar{A} \cap V_{\bar{\theta}}, G \cap \bar{\theta} \rangle$$

witness the $(\dot{A} \oplus \bar{A} \oplus G)$ -extendibility of κ . The assumption on $\bar{\lambda}$ ensures that $A \cap V_\lambda$ and $A \cap V_\theta$, where $\theta = j(\lambda)$, are definable respectively in these two structures. Thus, $j \upharpoonright V_\lambda : \langle V_\lambda, \in, A \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, A \cap V_\theta \rangle$ is an elementary embedding witnessing this instance of the A -extendibility of κ . So κ is A -extendible, and we have therefore proved that every class A in $V[G]$ admits an A -extendible cardinal. By theorem 6, therefore, $V[G]$ is a model of the Vopěnka principle, as desired. \square

It seems likely to me that the stretchability idea will be central in future A -extendibility lifting arguments, as it provides an extendibility analogue for the master condition technique.

Corollary 13. *The two Vopěnka axioms have exactly the same first-order consequences in the language of set theory. Specifically, GBC plus the Vopěnka principle proves a first-order statement ϕ in the language of set theory if and only if ZFC plus the Vopěnka scheme proves ϕ .*

$$\text{GBC} + \text{VP} \vdash \phi \quad \text{if and only if} \quad \text{ZFC} + \text{VS} \vdash \phi$$

In other words, GBC plus the Vopěnka principle is conservative over ZFC plus the Vopěnka scheme.

Proof. Since the Vopěnka principle implies the Vopěnka scheme, the converse direction is immediate. For the forward implication, theorem 12 shows that every model of ZFC plus the Vopěnka scheme can be expanded to a model of GBC plus the Vopěnka principle, with the same first-order part. So any first-order statement that fails in some model of ZFC plus the Vopěnka scheme also fails in some model of GBC plus the Vopěnka principle. So if a first-order statement is provable in

GBC plus the Vopěnka principle, then it must hold in all models of ZFC plus the Vopěnka scheme, and hence it is provable in that theory. \square

In particular, if one of the theories proves a contradiction, then so does the other, and therefore:

Corollary 14. *GBC plus the Vopěnka principle is equiconsistent with ZFC plus the Vopěnka scheme.*

REFERENCES

- [Ham02] Joel David Hamkins. A class of strong diamond principles. *preprint arxiv:math/0211419*, 2002.
- [Ham10] Joel David Hamkins. Can vopenka’s principle be violated definably? MathOverflow answer, 2010. URL:<http://mathoverflow.net/q/46538> (version: 2010-11-18).
- [Kan04] Akihiro Kanamori. *The Higher Infinite, Corrected Second Edition*. Springer-Verlag, 2004.

(J. D. Hamkins) MATHEMATICS, PHILOSOPHY, COMPUTER SCIENCE, THE GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 & MATHEMATICS, COLLEGE OF STATEN ISLAND OF CUNY

E-mail address: jhamkins@gc.cuny.edu

URL: <http://jdh.hamkins.org>